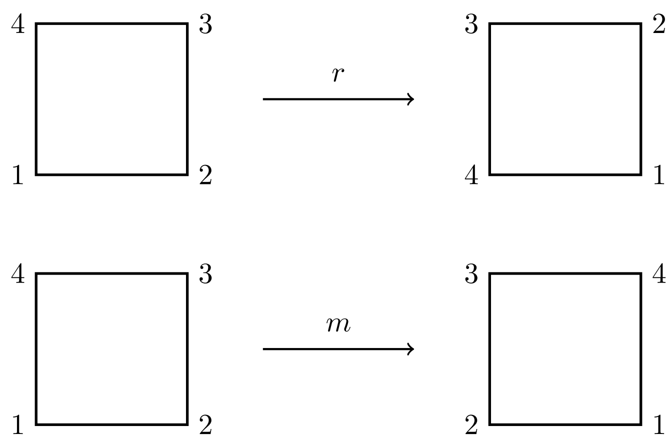
# Math 31 - Homework 3

Due Wednesday, July 10

**Note:** Any problem labeled as "show" or "prove" should be written up as a formal proof, using complete sentences to convey your ideas.

## Easier

**1.** Let $D_4$ be the 4th dihedral group, which consists of symmetries of the square. Let $r \in D_4$ denote counterclockwise rotation by $90°$, and let $m$ denote reflection across the vertical axis.




Check that

$$rm = mr^{-1}.$$

Conclude that $D_4$ is a nonabelian group of order 8.

**2.** We mentioned in class that elements of $D_n$ can be thought of as permutations of the vertices of the regular $n$-gon. For example, the rotation $r$ of the square mentioned in the last problem can be identified with the permutation

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Write the reflection $m$ as a permutation $\mu \in S_4$, and compute the product $\rho\mu$ in $S_4$. Then compute $rm \in D_4$, and write it as a permutation $\sigma$. Check that $\sigma = \rho\mu$. (In other words, this identification of symmetries of the square with permutations respects the group operations.)

**3.** Recall that if $*$ is a binary operation on a set $S$, an element $x$ of $S$ is an **idempotent** if $x * x = x$. Prove that a group has exactly one idempotent element.

**4.** Consider the group $\langle \mathbb{Z}_{30}, +_{30} \rangle$ under addition.

(a) Find the orders of the elements 3, 4, 6, 7, and 18 in $\mathbb{Z}_{30}$.

(b) Find all the generators of $\langle \mathbb{Z}_{30}, +_{30} \rangle$.

**5.** Determine whether each of the following subsets is a subgroup of the given group. If not, state which of the subgroup axioms fails.

(a) The set of real numbers $\mathbb{R}$, viewed as a subset of the complex numbers $\mathbb{C}$ (under addition).

(b) The set $\pi\mathbb{Q}$ of rational multiples of $\pi$, as a subset of $\mathbb{R}$.

(c) The set of $n \times n$ matrices with determinant 2, as a subset of $\mathrm{GL_n}(\mathbb{R})$.

(d) The set $\{i, m_1, m_2, m_3\} \subset D_3$ of reflections of the equilateral triangle, along with the identity transformation.

## Medium

**6.** [Saracino, Section 4, #25] Show that if $G$ is a finite group and $|G|$ is even, then there is an element $a \in G$ such that $a \neq e$ and $a^2 = e$.

**7.** [Saracino, Section 4, #21] Let $a$ and $b$ be elements of a group $G$. Show that if $ab$ has finite order $n$, then $ba$ also has order $n$.

**8.** [Saracino, Section 4, #20] Let $G$ be a group and let $a \in G$. An element $b \in G$ is called a *conjugate* of $a$ if there exists an element $x \in G$ such that $b = xax^{-1}$. Show that any conjugate of $a$ has the same order as $a$.

**9.** Let $G$ be a group. If $H$ and $K$ are subgroups of $G$, show that $H \cap K$ is also a subgroup of $G$.

**10.** Let $r$ and $s$ be positive integers, and define

$$H = \{nr + ms : n, m \in \mathbb{Z}\}.$$

(a) Show that $H$ is a subgroup of $\mathbb{Z}$.

(b) We saw in class that every subgroup of $\mathbb{Z}$ is cyclic. Therefore, $H = \langle d \rangle$ for some $d \in \mathbb{Z}$. What is this integer $d$? Prove that the $d$ you've found is in fact a generator for $H$.